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Exponential Solutions of Second-Order Systems

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EXPONENTIAL SOLUTIONS OF SECOND-ORDER SYSTEMS

by

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Abstract

It is shown for a 2 x 2 matrix depending analytically on a parameter t that the vanishing of all Lie functionals of an order exceeding a given but arbitrarily large number implies the vanishing of the second-order Lie functional. Some relations of Hellman for the coefficients of a matrix which is analytic in t and has certain vanishing Lie functionals are generalized.

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1. Introduction

In a previous paper[1] Magnus has given a representation for the fundamental solution of a general system of linear differential equations $\frac{dU}{dt} = AU$, $U(0) = I$, where $A = A(t)$ is a matrix and I is the unit matrix. This solution has the form $U = e^{\Omega}$, where $\Omega = B_1 + B_2 + B_3 + \dots$ and the B_i are Lie integral functionals. The importance of this representation is emphasized by Hellman's example[2] from which it appears that non-trivial systems of linear differential equations (i.e., systems that cannot be transformed into a system with a triangular matrix $A(t)$ of coefficients) have a fundamental solution $U = e^{\Omega}$ with Ω containing only a finite number of terms.

It is clearly important to know under what conditions Ω contains a finite number of terms. A general sufficiency criterion for this was given in [1].

In the present paper it is shown that if $A(t)$ is an analytic 2 x 2 matrix, then either $\Omega = \int_0^t A\,dt$ or Ω consists of infinitely many terms.

2. Definitions and Notation

Matrices are denoted by capitals; lower case letters stand for scalars. t is a real parameter. We shall write:

$$\int a = \int_c^t a(t)dt,$$

$$A = A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix}$$

$$\int A = \int_c^t Adt = \begin{pmatrix} \int_c^t a_{11}(t)dt & \int_c^t a_{12}(t)dt \\ \int_c^t a_{21}(t)dt & \int_c^t a_{22}(t)dt \end{pmatrix},$$

$$\Delta = \Delta(t) = a_{11}(t) - a_{22}(t),$$

$$[A,B] = AB - BA,$$

$$\mathrm{LIF}^{(2)} = \int \left[A, \int A\right],$$

$$\mathrm{LIF}_1^{(3)} = \int \left[\int A, \left[A, \int A\right]\right],$$

$$\mathrm{LIF}_2^{(3)} = \int \left[A, \int \left[A, \int A\right]\right].$$

$\mathrm{LIF}^{(n)}$ denotes Lie Integral Functional of weight n (see [1] for definition).

3. Condition for the termination of the sequence of Lie functionals

Assumption: Throughout this section we assume that A is a 2 x 2 matrix with analytic entires. Now we shall prove

Lemma 1. In order that $\mathrm{LIF}^{(2)} \equiv 0$ it is necessary and sufficient that two of the following six conditions holds:

$\Delta = 0$, $a_{12} = 0$, $a_{21} = 0$, $\Delta = \mathrm{const.} \cdot a_{12} \not\equiv 0$,

$\Delta = \mathrm{const.} \cdot a_{21} \not\equiv 0$, $a_{12} = \mathrm{const.} \cdot a_{21} \not\equiv 0$.

For proof it suffices to note that $\mathrm{LIF}^{(n)} \equiv 0$ is equivalent to $\frac{d}{dt}\mathrm{LIF}^{(n)} \equiv 0$ and that the entries d_{ij} of $\left[A, \int A\right]$ are:

$$d_{11} = a_{12}\int a_{21} - a_{21}\int a_{12},$$

$$d_{12} = \Delta\int a_{12} - a_{12}\int \Delta,$$

$$d_{21} = -\left(\Delta \int a_{21} - a_{21} \int \Delta\right)$$

$$d_{22} = -d_{11}.$$

Now let $\int B$ be a Lie integral functional of arbitrary weight $n \geq 2$, and let

$$B = \begin{pmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{pmatrix} .$$

Then the following lemma holds.

Lemma 2. The following four statements are incompatible:

(α) $B \not\equiv 0$, $\left[A, \int B\right] \equiv 0$, $\left[\int A, B\right] \equiv 0$, $\left[A, \int A\right] \equiv 0$.

Proof. Clearly

$$\left[A, \int B\right] = \begin{pmatrix} a_{12} \int b_{21} - a_{21} \int b_{21} & \Delta_a \int b_{12} - a_{12} \int \Delta_b \\ -\left(\Delta_a \int b_{21} - a_{21} \int \Delta_b\right) & a_{21} \int b_{12} - a_{12} \int b_{21} \end{pmatrix},$$

where

$$\Delta_a = a_{11} = a_{22}, \quad \Delta_b = b_{11} - b_{22},$$

and

$$\left[\int A, B\right] = \begin{pmatrix} \int a_{12} \cdot b_{21} - \int a_{21} \cdot b_{12} & \int \Delta_a \cdot b_{12} - \int a_{12} \cdot \Delta_b \\ -\left(\int \Delta_a \cdot b_{21} - \int a_{21} \cdot \Delta_b\right) & \int a_{21} \cdot b_{12} - \int a_{12} \cdot b_{21} \end{pmatrix}.$$

Assume that none of the six conditions in Lemma 1 holds and that (α) holds. Then

$$\text{(a)} \quad \int b_{21} = \int b_{12} \cdot \frac{a_{21}}{a_{12}} ,$$

$$\text{(b)} \quad b_{21} = b_{12} \cdot \frac{\int a_{21}}{\int a_{12}}$$

$$\text{(c)}\quad \int b_{12} = \int \Delta_b \cdot \frac{a_{12}}{\Delta_a},$$

$$\text{(d)}\quad b_{12} = \Delta_b \cdot \frac{\int a_{12}}{\int \Delta_a},$$

$$\text{(e)}\quad \int \Delta_b = \int b_{21} \cdot \frac{\Delta_a}{a_{21}},$$

$$\text{(f)}\quad \Delta_b = b_{21} \cdot \frac{\int \Delta_a}{\int a_{21}}.$$

$B \not\equiv 0$ implies in our case $b_{21} \not\equiv 0$, $b_{12} \not\equiv 0$, $b_{11} = -b_{22} \not\equiv 0$. Using (a) and (b) we have:

$$b_{12} \cdot \frac{\int a_{21}}{\int a_{12}} = \left(\int b_{12} \cdot \frac{a_{21}}{a_{12}}\right)' = \frac{a_{12}\left(b_{12}a_{21} + \int b_{12} \cdot a'_{21}\right) - a'_{12}\left(\int b_{12} \cdot a_{21}\right)}{a_{12}^2}$$

$$= \frac{a_{21}}{a_{12}} \cdot b_{12} + \left(\frac{a_{21}}{a_{12}}\right)' \int b_{12}.$$

Hence

$$\left(\frac{\int a_{21}}{\int a_{12}} - \frac{a_{21}}{a_{12}}\right) b_{12} = \left(\frac{a_{21}}{a_{12}}\right)' \int b_{12}$$

and

$$\int b_{12} = c \cdot \exp\left\{ \frac{a_{12}a'_{21} - a_{21}a'_{12}}{a_{12}\int a_{21} - a_{21}\int a_{12}} \cdot \frac{\int a_{12}}{a_{12}} \right\},$$

where c is a constant, $c \neq 0$. Using (c), (d), (e) and (f), we get analogous expressions for the other entries of $\int B$. Each of these expressions supplies the desired contradiction in the case under consideration. Likewise a contradiction is arrived at if we assume that one of the six conditions of Lemma 1 holds.

As a direct consequence of Lemma 2 we have

Theorem 1. For all $2x2$ Lie integral functionals of a given weight to vanish it is necessary and sufficient that $LIF^{(2)} \equiv 0$.

Corollary (cf. [1]). In the solution $U = e^{\Omega}$ of $\frac{dU}{dt} = AU$, $U(0) = I$, where A is a $2x2$ matrix with analytic entries, either

$$\Omega = \int_0^t A\,dt$$

or Ω is the sum of infinitely many terms.

Remark. Observe that the entries q_{ij} of $\left[A, \int \left[A, \int A\right]\right]$ are

$$q_{11} = -\left\{ a_{12} \cdot \int \left(\Delta \int a_{21} - a_{21} \int \Delta\right) + a_{21} \cdot \int \left(\Delta \int a_{12} - a_{21} \int \Delta\right)\right\},$$

$$q_{12} = \Delta \cdot \int \left(\Delta \int a_{12} - a_{12} \int \Delta\right) - 2a_{12} \cdot \int \left(a_{12} \int a_{21} - a_{21} \int a_{12}\right),$$

$$q_{21} = \Delta \cdot \int \left(\Delta \int a_{21} - a_{21} \int \Delta\right) + 2a_{21} \cdot \int \left(a_{12} \int a_{21} - a_{21} \int a_{12}\right),$$

$$q_{22} = -q_{11},$$

and that the entries p_{ij} of $\left[\int A, \left[A, \int A\right]\right]$ are

$$p_{11} = -\left\{\int a_{12} \cdot \left(\Delta \int a_{21} - a_{21} \int \Delta\right) + \int a_{21} \cdot \left(\Delta \int a_{12} - a_{12} \int \Delta\right)\right\},$$

$$p_{12} = \int \Delta \circ \left(\Delta \int a_{12} - a_{12} \int \Delta\right) - 2 \int a_{12} \cdot \left(a_{12} \int a_{21} - a_{21} \int a_{12}\right),$$

$p_{22} = - p_{11}$.

It is easy to see that if one of the six conditions in Lemma 1 holds then either $\mathrm{LIF}^{(2)} \equiv \mathrm{LIF}_1^{(3)} \equiv \mathrm{LIF}_2^{(3)} \equiv 0$ or $\mathrm{LIF}^{(2)} \not\equiv 0$, $\mathrm{LIF}_1^{(3)} \not\equiv 0$, $\mathrm{LIF}_2^{(3)} \not\equiv 0$. Now assume that none of the six conditions in Lemma 1 holds. Then for $\mathrm{LIF}_1^{(3)}$ to vanish identically it is necessary and sufficient that

$$\int \Delta = \pm 2 \sqrt{- \int a_{12} \cdot \int a_{21}} \ ,$$

or, equivalently, that $\int A$ have equal eigenvalues. At the same time $\mathrm{LIF}_2^{(3)} \not\equiv 0$.

Proof of Theorem 1. It suffices to consider the equations

$$(\beta) \int \Delta \cdot \left(\Delta \int a_{12} - a_{12} \int \Delta \right) - 2 \int a_{12} \cdot \left(a_{12} \int a_{21} - a_{21} \int a_{12} \right) = 0,$$

$$\int \Delta \cdot \left(\Delta \int a_{21} - a_{21} \int \Delta \right) + 2 \int a_{21} \cdot \left(a_{12} \int a_{21} - a_{21} \int a_{12} \right) = 0.$$

Putting $P = \int a_{12}$, $Q = \int a_{21}$, $Y = \int \Delta$ we can write these equations as follows:

$$(\gamma) \quad \begin{aligned} Y' Q - Q' Y &= - 2 Q V' / Y \\ Y' P - P' Y &= 2 P V' / Y, \end{aligned}$$

where $V = \int (P' Q - Q' P)$.

Multiplying the first of the equations (γ) by P and the second by Q and adding, we get

$$(1) \qquad 2P Q Y' - (PQ)' Y = 0.$$

Multiplying the first of the equations (γ) by ($-P'Y$) and the second by $Q'Y$, adding, and dividing the result by V', we get

$$(2) \qquad Y Y' + 2(PQ)' = 0 .$$

Now (1) implies $(PQ)' = 2PQY'/Y$. (2) implies $(PQ)' = -Y Y'/2$. Hence $Y^2 = -4PQ$ or $Y = \pm 2\sqrt{-PQ}$. Substitution of either value for Y in the equations (β) shows that these values actually satisfy (β). The second part of our statement can be proved directly or inferred from Theorem 1.

4. A note on Hellman's relations

For what follows capital letters denote constant (complex) $n \times n$ matrices.

Definition. $P^{(0)} = P$, $P^{(k)} = \left[A_0, P^{(k-1)}\right]$, $k = 0, 1, 2, \ldots$

We note the following relations:

$$(A) \qquad [B,C]^{(k)} = \sum_{j=0}^{k} \binom{k}{j} \left[B^{(j)}, C^{(k-j)}\right],$$

$$(B) \qquad \left[D_1 \left[D_2 \left[\cdots \left[D_q, D_{q+1}\right] \cdots \right]^{(k)} = \right.\right.\right.$$

$$= \sum_{j_1=0}^{k} \sum_{j_2=0}^{k-j_1} \cdots \sum_{j_q=0}^{k-j_1 \cdots j_{q-1}} \binom{k}{j_1} \binom{k-j_1}{j_2} \cdots \binom{k-\sum_{p=1}^{q-1} j_p}{j_q}$$

$$\left[D_1^{(j_1)} \left[D_2^{(j_2)} \left[\cdots \left[D_q^{(j_q)}, D_{q+1}^{(k-\sum_{p=1}^{q} j_p)}\right] \cdots \right]\right.\right.\right. .$$

(C) Let $B^{(m)} = 0$, $C^{(n)} = 0$, $0 < n+m$, $0 \leq n \leq m$. Then

$$[B,C]^{(m+n-1)} = \left[B^{(0)},C^{(m+n-1)}\right] + \dots + \alpha\left[B^{(m-1)},C^{(n)}\right] + \beta\left[B^{(m)},C^{(n-1)}\right]$$
$$+ \dots + \left[B^{(m+n-1)},C^{(0)}\right] = 0.$$

(D) $D_j^{(m_j)} = 0$ (for $j = 1, 2, 3, \dots$) implies

$$\left[D_1 \left[\dots\left[D_q,D_{q+1}\right]\dots\right]^{(m_1+\dots+m_{q+1}-q)} = 0.$$

(If $D_{j_0} = A_0$ then m_{j_0} in $(m_1 + \dots + m_{q+1}-q)$ should be 0.) This last result is independent of the distribution of brackets in the commutator.

We now apply the above to give a simple proof of a result proved by M. Hellman[2].

Let

$$\sum_{\substack{p+q=k \\ 0 \leq p < q}} \alpha_{pq} \left[A_p,\ A_q\right] = 0, \qquad k = 1, 2, \dots; \ \alpha_{pq} \neq 0.$$

Then

$$\alpha_{0k} \left[A_0\ A_k\right] = -\alpha_{1,k-1} \left[A_1,A_{k-1}\right] - \alpha_{2,k-2} \left[A_2,A_{k-2}\right] - \dots .$$

Since

$$\left[A_0,A_1\right] = \left[A_0,A_2\right] = A_2^{(1)} = 0$$

and since we may assume (by induction) that

$$A_{k-j}^{(k-j-1)} = 0, \qquad j = 1, \dots, k-2; \ (k-j \geq 2),$$

it follows (cf. (C), (D)) that

$$A_k^{(k-1)} = 0, \quad k = 2, 3, \ldots$$

or better still,

$$A_k^{(\min\left\{k-1, 2\alpha-1\right\})} = 0,$$

since no elementary division of A_0 is of higher degree than α. (see Roth[3] Theorem 5). Now if we assume that

$$A = A(t) = \sum_{t=0}^{\infty} A_i t^i,$$

then the coefficients of $\left[A, \int_0^t A\, dt\right]$ and of $\int_0^t \left[A, \int_0^t A\, dt\right]$ are of the form considered above. Hence

$$\int_0^t \left[A, \int_0^t A\right] \equiv 0$$

implies

$$A_k^{(k-1)} = 0, \quad k = 2, 3, \ldots,$$

which is the content of Hellman's result[2].

This result can be extended to certain Lie integral functionals of arbitrary weight. Thus let

$$LIF_1^{(1+1)} = \int_0^t \left[A, \int_0^t A\, dt\right]$$

and

$$LIF_1^{(s+1)} = \int_0^t \left[A, LIF_1^{(s)}\right],$$

where

$$A = \sum_{i=0}^{\infty} A_i \, t^i.$$

Then

$$LIF_1^{(s+1)} = \sum_{i=1}^{\infty} B_i \, t^{s+i+1},$$

where

$$B_p = \alpha_p A_p^{(s)} + C_p, \qquad \alpha_p \neq 0, \; p = 1, 2, \ldots ;$$

here $C_1 = 0$, and C_p is a linear combination of terms of the form

$$\left[A_{j_1}\left[A_{j_2}\left[\ldots\left[A_{j_s}, A_{j_{s+1}}\right]\ldots\right]\right]\right]$$

with $j_1 + \ldots + j_{s+1} = p$, $j_n < p$, $n = 1, \ldots, s+1$, $j_s < j_{s+1}$.

Now $LIF_1^{(s+1)} \equiv 0$ implies $B_p = 0$ for all p. In particular $A_1^{(s)} = 0$. Assume $A_m^{(sm)} = 0$, $1 < m < p$. By (D) above,

$$\left[A_{j_1}\left[A_{j_2}\left[\ldots\left[A_{j_s}, A_{j_{s+1}}\right]\ldots\right]\right]\right]^{s(p-1)} = 0.$$

Hence $A_p^{(sp)} = 0$ for $p = 1, 2, \ldots$.

The case $s = 1$ yields a stronger result than the case of a general s. The reason becomes clear from the following formulation of our result.

Theorem 2. If in $LIF_1^{(s+1)} = B_1 t^{s+2} + B_2 t^{s+3} + \ldots \equiv 0$, B_f is the last coefficient consisting of the single term $A_f^{(s)}$, then

$$A_{f+k}^{((k+1)s)} = 0, \qquad k = 0, 1, 2, \ldots ,$$

or better still, by the result of Roth[3] referred to above,

$$A_{f+k}^{(\min\{(k+1)s,\ 2\alpha-1\})} = 0.$$

Corollary (cf. [1]). If in $\frac{dU}{dt} = AU$, $U(0) = I$ (the unit matrix), A has analytic entries, then the validity of the relations of Theorem 2 is necessary for Ω to consist of a finite number of terms only.

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